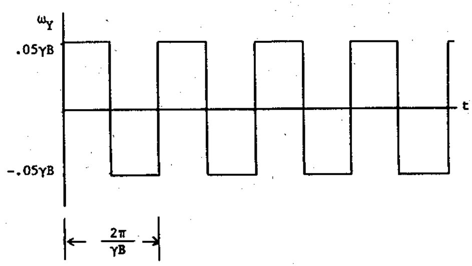

Fig. 1 Simulated oscillatory Y-axis rate.

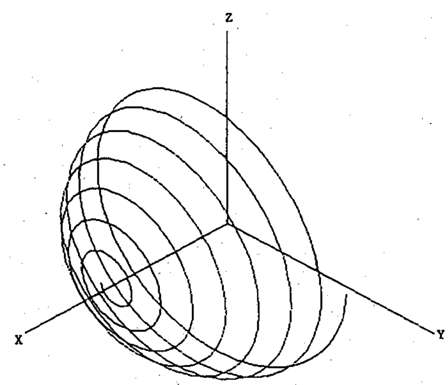

Fig. 2 Effect on X-axis gyro of oscillatory rate about Y-axis.

about a cross axis and at a frequency corresponding to the Larmor frequency, a periodic loss of magnetometer output results. This was demonstrated with a digital simulation of the ^{3}He gyro dynamics. The particular case demonstrated was an X-axis gyro in response to an oscillating rate about the Y axis. The cross-axis rate input (Fig. 1) is described by

$$\omega_Y = 0.05\gamma B \operatorname{sgn}[\cos\gamma Bt] \tag{6}$$

The resulting path of the magnetization vector, as seen in the gyro frame, is shown to wind its way up the gyro X axis (Fig. 2). The magnitude of M is unchanged since relaxation effects are ignored, but if the oscillatory input persists, M winds back down into the $X=0$ plane and then continues its way down the $-X$ axis and back and forth. Since the magnetometer senses the component of M orthogonal to B the effect is to alternately diminish and restore the magnetometer signal. Overall then, the signal-to-noise ratio of the instrument is decreased. In this demonstration time has been scaled in terms of γB. For example, if γB is shosen to be 2π rad/s, then ω_Y has a peak rate of 0.31 rad/s and the total time shown (Fig. 2) is about 7.5 s. In other words, if the peak input rate is 5% of the Larmor rate, it will take about 7.5 Larmor periods for M to wind its way out of the $X=0$ plane.

The situation is most easily explained in a rotating frame. For example, approximate ω_Y as

$$\omega_Y = 0.05\gamma B \cos\gamma Bt \tag{7}$$

Then represent ω_Y as two counter-rotating vectors in the $X=0$ plane, each of magnitude 0.25 γB, starting in the Y-axis direction and rotating at γB rad/s. Then, in the rotating frame, the vector traveling with the rotating frame is stationary, while the other appears to rotate at twice the Larmor rate. The effects of the latter average to zero, but the former causes M to precess about the rotating frame Y axis. In the gyro frame both the Y axis motion and the Larmor precession about the X axis cause the spiral trajectory shown. This situation is a worst-case situation. For oscillatory rates much higher or lower than γB, the effect averages to zero.

Conclusions

Cross-axis rates, oscillating at the Larmor frequency, will cause a degradation in the signal-to-noise ratio of nuclear magnetic resonance gyros. This has been demonstrated for a single-species, unpumped device but should cause related problems in the dual-species, continuously pumped devices as well. Care must be taken with such instruments to isolate them from vibrations at the Larmor frequency.

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An Exact Expression for Computing the Degree of Controllability

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Introduction

THE concept of degree of controllability was recently introduced to study control systems associated with large flexible spacecraft.[1-5] However, a formula for the exact value of the degree of controllability was not determined; instead various techniques for computing an estimate were developed in Refs. 1-5. Here we give an exact formula which can be used to compute the degree of controllability.

In the next section, we review the concepts and results associated with the idea of degree of controllabiltiy and then derive the above mentioned result. In a subsequent section, this result is used to compute the degree of controllability for several of the examples in Refs. 1-5.

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Some Controllability Concepts and Results

Consider a linear time-invariant system

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad t\epsilon[0,\infty) \tag{1}$$

where $x(t)\epsilon R^n$ is the *state* and $u(t)\epsilon R^m$ is the *control* having components $u_i(\cdot)$ which are required to be measurable and are restricted to lie in a prespecified *constraint set* Ω given by

$$\Omega = \{u: \ |u_i| \leq 1, \qquad i=1,2,...m\} \tag{2}$$

Let $M(\Omega)$ denote the set of functions from R into Ω that are measurable on $[0,\infty)$. If $u(\cdot)\epsilon M(\Omega)$, then $u(\cdot)$ is termed *admissible.* Given an initial state x_0 and an admissible control $u(\cdot)$, we denote the corresponding solution of Eq. (1) by $x(t,x_0,u(\cdot))$.

Definition 1. The system [Eq. (1)] is Ω-null controllable from x_0 at time $T\geq 0$ if there exists an admissible control such that $x(T,x_0,u(\cdot))=0$.

Definition 2. The recovery region at time T is the set

$$R(T) \triangleq \{x_0\epsilon R^n: \ x(T,x_0,u(\cdot)) = 0 \text{ for some } u(\cdot)\epsilon M(\Omega)\}$$

Definition 3. The degree of controllability in time T is defined as

$$\rho(T) \triangleq \inf\{\|x(0)\|: \ x(0)\epsilon R^c(T)\}$$

where $R^c(T)$ is the complement of $R(T)$.

The recovery region is the set of initial states which can be steered to the origin at time T by an admissible control, and the degree of controllability at T is a measure of the size of the recovery region at T. In fact, $\rho(T)$ is the radius of the largest ball, centered at the origin, that is contained in $R(T)$.

To obtain an exact expression for the degree of controllability $\rho(T)$, we need the following theorem from Refs. 6-7.

Theorem 1. The system [Eq. (1)] is system is Ω-null controllable from x_0 at T if and only if

$$\min_{\|z_0\|=1}\left[x_0'z_0 + \int_0^T \max_{\omega\epsilon\Omega}(\omega' B' z(\tau))\,\mathrm{d}\tau\right] \geq 0$$

where $z(\cdot)$ is a solution of the adjoint system

$$\dot{z}(t) = -A'z(t), \qquad z(0) = z_0$$

We now present our main result.

Theorem 2. The degree of controllability in time T is

$$\rho(T) = \min_{\|z_0\|=1}\left[\int_0^T \max_{\omega\epsilon\Omega}(\omega' B' z(\tau))\,\mathrm{d}\tau\right] \tag{3}$$

Proof. Given $r>0$, let $B_r(0) \triangleq \{x_0\epsilon R^n: \ \|x_0\| \leq r\}$. Then, from Theorem 1, every $x_0\epsilon B_r(0)$ is Ω-null controllable at T if and only if for all $x_0\epsilon B_r(0)$

$$\min_{\|z_0\|=1}\left[x_0'z_0 + \int_0^T \max_{\omega\epsilon\Omega}(\omega' B' z(\tau))\,\mathrm{d}\tau\right] \geq 0$$

or, equivalently,

$$\inf_{x_0\epsilon B_r(0)} \min_{\|z_0\|=1}\left[x_0'z_0 + \int_0^T \max_{\omega\epsilon\Omega}(\omega' B' z(\tau))\,\mathrm{d}\tau\right] \geq 0 \tag{4}$$

Interchanging the order of inf and min and using the fact that

$$\inf\{x_0'z_0: \ x_0\epsilon B_r(0)\} \not\equiv r\|z_0\| = -r$$

we have from Eq. (4) that every $x_0\epsilon B_r(0)$ is Ω-null controllable at T if and only if

$$\min_{\|z_0\|=1}\left[\int_0^T \max_{\omega\epsilon\Omega}(\omega' B' z(\tau))\,\mathrm{d}\tau\right] \geq r \tag{5}$$

Sine the largest value of r for which Eq. (5) holds is the degree of controllability, the theorem is proved.

To compute the degree of controllability, a finite-dimensional optimization problem needs to be solved. For simple problems, this can be done directly. For more complicated systems, numerical tehcniques are needed. Since the computation involves only a finite-dimensional optimization problem, the expression for $\rho(T)$ is particularly suitable for numerical procedures. Note that the minimization in Eq. (3) is over a function that may not be differentiable, and a derivative-free algorithm must be used.

Examples

In Ref. 3, three second order examples were considered—the harmonic oscillator, the damped harmonic oscillator and the double integral plant. For each of the examples, the exact degree of controllability can be determined analytically and the exact value was compared to the value of $\rho(T)$ found using several approximation techniques. To illustrate our technique, we consider one of these examples in detail and only present our results for the other two.

Example 1 (Harmonic Oscillator)

The system equations are

$$\dot{x}_1(t) = \Omega x_2(t), \qquad \dot{x}_2(t) = -\Omega x_1(t) + u(t)$$

and the control must satisfy $|u(t)| \leq 1$. For this system, the adjoint equations are

$$\dot{z}_1(t) = \Omega z_2(t), \qquad \dot{z}_2(t) = -\Omega z_1(t)$$

and their solution is

$$z_1(t) = z_{10}\cos\Omega t + z_{20}\sin\Omega t$$

$$z_2(t) = z_{20}\cos\Omega t - z_{10}\sin\Omega t$$

Using Theorem 2,

$$\rho(T) = \min_{\theta z_0|=1}\left[\int_0^T \max_{|\omega|\leq 1}(\omega z_2(\tau))\,\mathrm{d}\tau\right]$$

$$= \min_{\|z_0\|=1}\int_0^T |z_2(\tau)|\,\mathrm{d}\tau = \min_{\|z_0\|=1}\int_0^T |z_{20}\cos\Omega\tau - z_{10}\sin\Omega t|\,\mathrm{d}\tau$$

By using the constraint $z_{10}^2 + z_{20}^2 = 1$ to eliminate z_{20}, this two-dimensional minimization problem to determine $\rho(T)$ can be replaced with an equivalent one-dimensional problem. The resulting one-dimensional problem was solved on an Apple II+ by the "golden section search" technique as described in Ref. 8. Our results for $\omega=1$ are shown in Fig. 1 and agree with those in Ref. 3.

Example 2 (Damped Harmonic Oscillator)

With damping present, the system equations become

$$\dot{x}_1(t) = -\alpha x_1(t) + \frac{n_2\Omega}{n_1}x_2(t)$$

$$\dot{x}_2(t) = -\frac{n_1\Omega}{n_2}\int \dot{x}_2 = (t) - \alpha x_2(t) + \frac{1}{n_2}u(t)$$

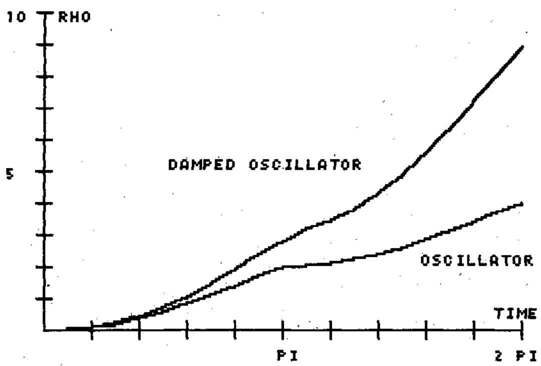

Fig. 1 Degree of controllability for Examples 1 and 2.

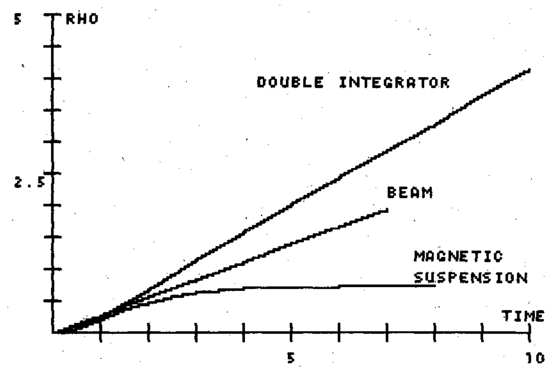

Fig. 2 Degree of controllability for Examples 3, 4, and 5.

The same technique as used for Example 1 is used here and our results are shown in Fig. 1 for $n_1 = n_2 = \omega = 1$, $\alpha = ¼$. Again, they agree with the results in Ref. 3.

Example 3 (Double Integral Plant)

The equations for this system are

$$\dot{x}_1(t) = x_2(t), \quad \dot{x}_2(t) = u(t)$$

with $|u(t)| \leq 1$. The results are shown in Fig. 2 and are the same as those in Ref. 3.

Example 4 (Magnetic Suspension)

In the three previous examples, $\rho(T)$, as a function of time, increases without bound. In this example, $\rho(T)$ approaches a finite limit as T becomes large. The system equations are

$$\dot{x}_1(t) = x_2(t), \qquad \dot{x}_2(t) = x_1(t) + u(t)$$

These equations describe a magnetic suspension system (see Ref. 9 for details). The same numerical procedure as used in Examples 1-3 was used on this problem and the results are shown in Fig. 2. Note that for $T>6$, $\rho(T)$ is essentially constant.

Example 5 (Simply Supported Beam)

The most interesting example is that of Ref. 1 where the authors consider a simply supported beam with actuators placed at one-third and two-thirds the length of the beam. There, the authors could not compute the exact degree of controllability and only obtained the approximate values for $T \leq 4$.

The system is modeled by the equations

$$\dot{x}_1(t) = 2.47\, x_2(t)$$

$$\dot{x}_2(t) = -2.47\, x_1(t) + 0.866u_1(t) + 0.866u_2(t)$$

$$\dot{x}_2(t) = 9.87x_4(t)$$

$$\dot{x}_4(t) = -9.87x_3(t) + 0.216u_1(t) - 0.216u_2(t)$$

with control constraints $|u_1(t)| \leq 1$, $|u_2(t)| \leq 1$.

For this problem

$$\rho(T) = \min_{\|z_0\|=1} \left\{ \int_0^T [\,|h_1(\tau)| + |h_2(\tau)|\,] \mathrm{d}\tau \right\}$$

where

$$h_1(\tau) = 0.866(z_{20}\cos 2.47\tau - z_{10}\sin 2.47\tau)$$
$$+ 0.216(z_{40}\cos 9.87\tau - z_{30}\sin 9.87\tau),$$

$$h_2(\tau) = 0.866(z_{20}\cos 2.47\tau - z_{10}\sin 2.47\tau)$$
$$+ 0.216(z_{30}\sin 9.87\tau - z_{40}\cos 9.87\tau)$$

To solve this minimization problem, we used the derivative-free algorithm MINIMUM as described in Ref. 8. Our results for the *exact* degree of controllability are shown in Fig. 2.

In Ref. 1 a technique to determine the approximate degree of controllability is presented. In addition to only providing approximate values of $\rho(T)$, it involves discretization and is more difficult to implement than the technique used here based on the exact expression for $\rho(T)$. Furthermore, the accuracy of the approximate technique depends on the step size used in discretizing the system, and an appropriate step size may have to be determined by numerical experimentation.

Concluding Remarks

The degree of controllability is a measure of the size of the set of initial states that can be steered to the region in a prescribed time. An exact expression for the degree of controllability has been obtained for a constant, linear system when there are magnitude constraints on the control and the target is the origin. Several examples were presented to illustrate how this expression could be used to compute the degree of controllability.

Acknowledgment

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